

Real time dynamics and phase separation in a holographic first order phase transition

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We study the fully nonlinear time evolution of a holographic system possessing a first order phase transition. The initial state is chosen in the spinodal region of the phase diagram, and includes an inhomogeneous perturbation in one of the field theory directions. The final state of the time evolution shows a clear phase separation in the form of domain formation. The results indicate the existence of a very rich class of inhomogeneous black hole solutions.

Introduction. The concept of a phase transition is inherently an equilibrium one and it is a theoretical challenge to formulate a framework in which it can be quantitatively studied in cases involving real time dynamics. Such a framework is offered by gauge/gravity duality put forward about twenty years ago [1]. Gauge/gravity duality, also referred to as holography, is immanently related to field theory systems at strong coupling and thus it is very useful to study nonperturbative physics especially as within this framework one can work directly in Minkowski signature and study real time dynamics. Recently, it has been demonstrated that well known holographic 1st order phase transitions [2] are accompanied by an unstable spinodal region [3, 4] in accordance with standard predictions [5]. The analysis was based on linear response theory, however the true power of the dual gravitational approach is the possibility of investigating fully nonlinear dynamic evolution of the system. Simultaneously with the present work, this particular direction has been recently undertaken in Ref. [6] where spinodal region was studied in the holographic model of a phase transition. The final state of the time evolution was an inhomogeneous configuration approached at late times within the hydrodynamic approximation. A different set-up of a homogeneous evolution has been undertaken in Ref. [7]. Inhomogeneous, static configurations appearing in the context of a holographic 1st order phase transition were also recently studied in Ref. [8].

A natural question which arises is whether the final state will exhibit domains of the two coexisting phases with the same values of the free energy. The main result of the present letter is to demonstrate for the first time that in the case of a three-dimensional nonconformal system with a holographic dual such a phase separation will arise dynamically through a real time evolution from a perturbation in the spinodal region. The respective energy densities of the two components of the final state are very close to the corresponding energy densities determined at the critical temperature. This implies that the system undergoes a dynamical transition during which different regions of space become occupied with different phases of matter.

The holographic framework. The holographic model

we use is a bottom-up construction containing Einstein gravity coupled to a real, self-interacting scalar field. The action takes the standard form

$$S = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right] + S_{\text{GH}} + S_{\text{ct}}, \quad (1)$$

where κ_4 is related to the four dimensional Newton's constant $\kappa_4 = \sqrt{8\pi G_4}$, and the self interaction potential $V(\phi)$ is given below. We include the boundary terms in the form of Gibbons-Hawking S_{GH} [9] and holographic counterterms S_{ct} [10, 11] contributions. We chose to work in 3+1-dimensional bulk spacetime, which is dual to 2+1-dimensional field theory, for the absence of conformal anomaly in odd dimensions. This, in turn, makes the expansions near the conformal boundary free from logarithms, which allows for the usage of Chebyshev spectral methods for numerical integration. The $V(\phi)$ potential is constructed so that the dual field theory undergoes an equilibrium first order phase transition. The specific choice that we use is

$$V(\phi) = -6 \cosh\left(\frac{\phi}{\sqrt{3}}\right) + b_4 \phi^4, \quad (2)$$

where $b_4 = -0.2$. This functional form is dual to a relevant deformation of the boundary CFT with an operator of conformal dimension $\Delta = 2$. When $b_4 = 0$ the potential is that of $\mathcal{N} = 2$ supergravity in $D = 4$ after dimensional reduction from $D = 11$ [12]. The physical scale, breaking conformal invariance, is set by the source of the operator, and is chosen to be $\Lambda = 1$. The equilibrium structure of this model is described in terms of dual black hole geometries characterized by specifying the horizon value of the scalar field, i.e., $\phi(z=1) = \phi_H$. The entropy and the free energy of the system are, in turn, given by the Bekenstein-Hawking formula, and the on-shell value of the action (1) respectively. The corresponding thermodynamics reveals the appearance of a first order phase transition between different branches of black hole geometries as determined by the difference of free energies. The order of the transition is established by a discontinuity of the first derivative of the free energy of the system. This effect is illustrated in the lower panel of Fig. 1. The value of the critical temperature is

$T_c \simeq 0.246$ (in $\Lambda = 1$ units). This transition is of similar nature as the Hawking-Page transition in the case of AdS space [13, 14] (an important difference, however, is that in the current setup all phases are of the black hole type). The equation of state (EoS) is displayed in the upper panel of Fig. 1 as a temperature dependence of the energy density. This EoS is similar to the $4D/5D$ case studied in [3, 4] and the detailed analysis of the linearized dynamics proved, in accordance with the general lore, the existence of a spinodal region separating stable configurations [3, 4].

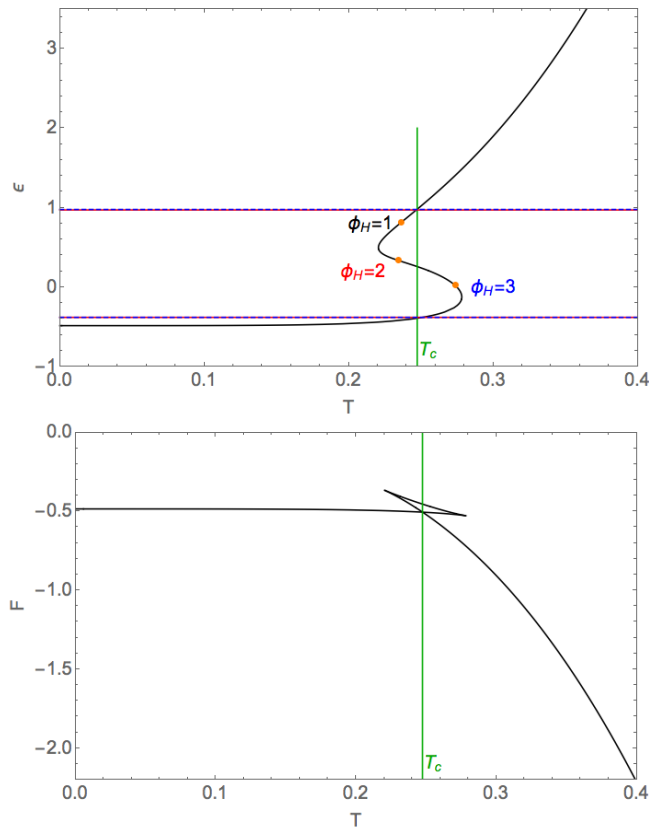


FIG. 1. Upper panel: The temperature dependence of the holographic energy density. Vertical green line represents the transition temperature. Orange points represent sample chosen initial configurations for time evolution. Horizontal lines show the domain energies in the final state (solid - cosine perturbation, dashed - Gauss perturbation). Lower panel: free energy as a function of temperature.

Time dependent configurations. To study the time evolution of the system we adopt the following metric ansatz in the Eddington-Finkelstein (EF) coordinates

$$ds^2 = -A dt^2 - \frac{2 dt dz}{z^2} - 2 B dt dx + S^2 (G dx^2 + G^{-1} dy^2), \quad (3)$$

where all functions are x, t, z dependent. We can take the initial state to lie in the spinodal region of the phase diagram and add an x -dependent perturbation to the S

function. By the proper choice of the perturbing function

$$\delta S(t, x, z) = S_0 z^2 (1 - z)^3 \cos(kx), \quad (4)$$

we can turn on a particular unstable mode or add a mixture of all the modes

$$\delta S(t, x, z) = S_0 z^2 (1 - z)^3 \exp\left(-w_0 \cos\left(\tilde{k}x\right)^2\right), \quad (5)$$

with different widths. By solving the time dependent Einstein-matter equations of motion with proper AdS boundary conditions at $z = 0$ ¹ we determine the non-linear evolution of the system. Using the procedure of holographic renormalization we then read-off the relevant observables like the energy density of the the boundary theory from subleading terms in the near-boundary expansion [10, 11].

The system is essentially studied in the microcanonical ensemble as the total energy density of the system is fixed throughout the evolution. The gravitational formulation of the problem is now given by a coupled set of non-linear partial differential equations. We solve that problem numerically using the characteristic formulation of General Relativity [15] along with spectral methods [16]. In the relevant spatial direction we use periodic boundary conditions with spectral Fourier discretization. The remaining spatial direction is uncompactified.

Results. Since for this system the energy density in equilibrium uniquely determines the temperature, we may deduce that the final state of evolution starting from the unstable spinodal branch necessarily has to be inhomogeneous. The physical expectation that the final state will consist of well separated phases at the phase transition temperature $T = T_c$ would manifest itself in the existence of spatial domains characterized by very flat energy density, the values of which should coincide with the energy densities of the two stable phases at $T = T_c$. The fact that such a configuration is attained dynamically even when starting from points on the spinodal branch with temperatures differing from T_c is far from trivial. This is the main result of the present work.

To illustrate the effect of the appearance of different phases during the time evolution we run the simulations for a number of initial configurations, covering a representative region of interest. Some of the configurations are marked with orange dots in Fig. 1. As it was explained in the previous section we use two different shapes of perturbing function given in Eq. (4) and (5) with different values for parameters. Particularly transparent results appear for the value of the momentum equal to $k = 1/6$ and $\tilde{k} = 1/12$, and we have chosen

¹ In the EF coordinates $A \sim 1/z^2 + O(1)$, $S \sim 1/z + O(1)$, $G \sim O(z)$, $B \sim O(z)$ for $z \rightarrow 0$.

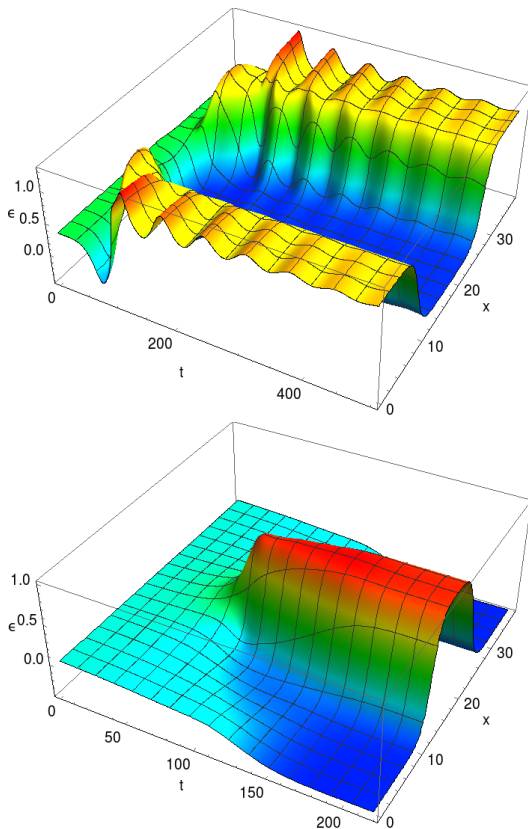


FIG. 2. The energy density as a function of time for the initial configuration in spinodal region: $\phi_H = 2$ Gaussian perturbation (upper panel), $\phi_H = 3$ cosine perturbation (lower panel).

to present those in this letter. The amplitudes of the perturbations are in the range $S_0 = 0.1 - 0.5$.

The first point of interest is a large black hole configuration with temperature below T_c , but still on the stable branch e.g. with $\phi_H = 1$. The linear analysis shows no instability of that configuration. However, one could still expect a non-linear instability due to overcooling. In our simulations we found, however, no evidence of that within the considered framework.

The second considered point, with $\phi_H = 2$, is placed deep in the unstable region. The temporal and spatial dependence of the energy density is shown in the upper panel of Fig. 2. Initially small Gaussian perturbation grows with time, and after around 400 units of simulation time starts settling down to an inhomogeneous final state. The maximum and the minimum energy of this state, marked as horizontal solid lines in Fig. 1, approach within less than 1% the energy densities determined by the transition temperature. Pronounced flat regions of constant energy density are apparent at late stages of the evolution. It is clearly visible that in the final state different parts of the system are occupied by the different phases, joined by a domain wall.

The third considered configuration, with $\phi_H = 3$, lies

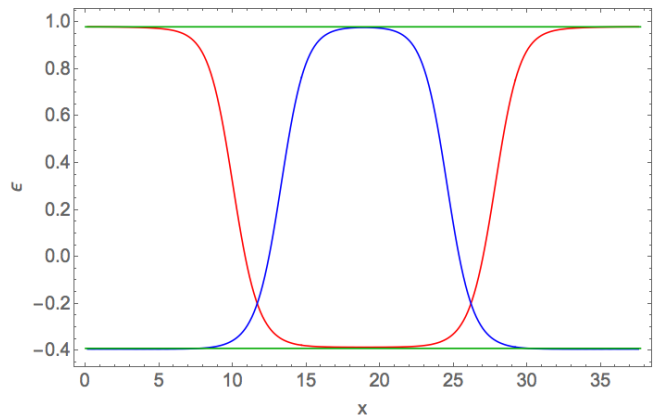


FIG. 3. The final state energy density for different initial configurations: $\phi_H = 2$ (red) and $\phi_H = 3$ (blue). The horizontal green lines represent the energy densities at the critical temperature in isotropic solutions.

close to the end of the spinodal region. The time dependence of the energy density in this case is displayed in the lower panel of Fig. 2. The perturbation added is a single cosine mode. In this case, after about 150 units of simulation time, the configuration settles down to an inhomogeneous final state. The maximum and the minimum energy densities of this final state are marked with horizontal, dashed lines in Fig. 1. Similarly to the previous configuration the extrema of energy density reach the corresponding densities determined at the transition temperature $T = T_c$. However, due to the fact that the initial energy density is smaller than the energy density of a configuration with $\phi_H = 2$, we observe a smaller region of the high-energy phase in the final state.

To summarize the above results we display the final state energy density as a function of x in Fig. 3 for both unstable initial configurations. For completeness we also show (Fig. 4) the time and spatial dependence of the expectation value of the scalar field. Both quantities display consistently a phenomenon of bubble formation triggered by a spinodal instability. Because the initial energy density is close to an average of both critical energy densities approximately half of the system is in the high energy phase and the other half is in the low energy phase. Both regions display very flat behaviour near the corresponding extremal points. For the other initial state, i.e., $\phi_H = 3$, the initial energy density is smaller and in turn the final state reveals a suppression of the high energy phase which occupies only a small fraction of the total (transverse) volume. Another notable difference, appearing with both types of perturbations, is that for the $\phi_H = 2$ case it takes longer for the system to reach a stable final configuration.

Discussion. In this letter we have demonstrated for the first time the existence of a phase separation effect at the transition temperature of a 1st order phase transition in

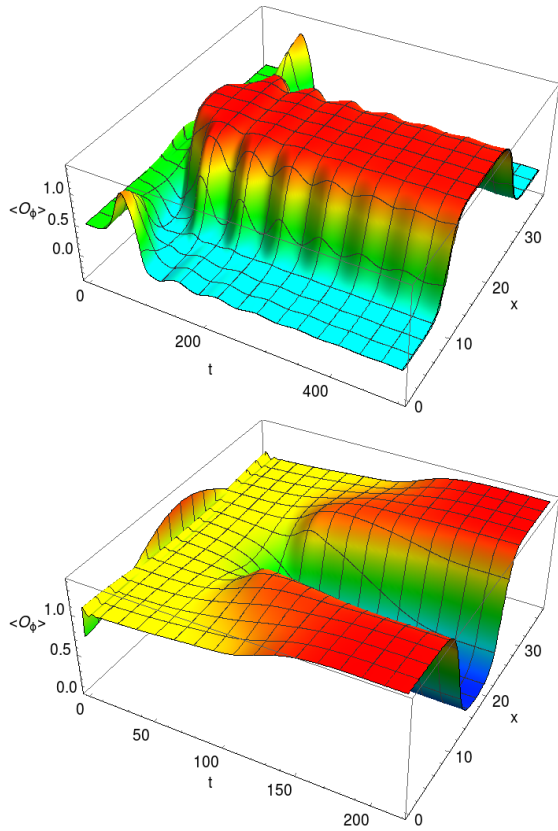


FIG. 4. The space-time evolution of the expectation value of scalar field in dual boundary theory in spinodal region: $\phi_H = 2$, Gaussian perturbation (upper panel) and $\phi_H = 3$ cosine perturbation (lower panel).

the context of holographic models. The chosen set-up is a bottom-up holographic construction designed to exhibit an equilibrium first order phase transition.

Full nonlinear time evolution of a perturbed configuration with a spinodal instability ends in an inhomogeneous state composed of domains of different stable phases as determined at the transition temperature fixed by the equality of free energies. The dual gravitational configuration is a black hole with an inhomogeneous horizon. However in contrast to the results of [6], the geometries obtained here correspond to domains of specific thermodynamic phases with very mild spatial dependence separated by relatively narrow domain walls. On the gravitational side this means that the bulk geometry consists of two distinct types of black holes, characterized e.g. by the value of the scalar field on the horizon smoothly connected by interpolating domain walls. From the field theory interpretation we expect to have an immense moduli space of geometries which correspond to different configurations of phase domains coming from different seed perturbations. Indeed, we found also solutions with multiple potential domains which, however, were of size of the order of the domain wall width and thus were more

similar to the solutions of [6]. It is clear that with an appropriately large transverse spatial extent of the simulation one can explicitly construct solutions with multiple domains. We intend to investigate this in the future.

We have also demonstrated that large black holes with temperatures below the critical value are stable against the perturbations that we tried. This indicates that in the holographic description, we may naturally expect an overcooled phase which only starts to nucleate once it has dynamically moved into the spinoidal branch.

There are numerous directions for further research. Apart from studying the space of domain geometries mentioned earlier, it would be very interesting to investigate in detail the various temporal regimes which can be seen in figures 3 and 4 and study the dynamics of phase domains. Naturally it would be good to investigate extensions to other dimensions as well as ultimately relax any symmetry assumptions. An additional bonus of the present setup is the appearance configurations breaking translation invariance without putting explicit inhomogeneous sources. In other words, the final state solutions spontaneously break translational invariance. This opens a possibility of applications in the context of condensed matter physics [17, 18].

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- [1] J. M. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231,
- [2] S. S. Gubser and A. Nellore, Phys. Rev. D **78**, 086007 (2008),
- [3] R. A. Janik, J. Jankowski and H. Soltanpanahi, Phys. Rev. Lett. **117**, no. 9, 091603 (2016)
- [4] R. A. Janik, J. Jankowski and H. Soltanpanahi, JHEP **1606**, 047 (2016)
- [5] P. Chomaz, M. Colonna and J. Randrup, Phys. Rept. **389**, 263 (2004),
- [6] M. Attems, Y. Bea, J. Casalderrey-Solana, D. Mateos, M. Triana and M. Zilhao,
- [7] U. Gürsoy, A. Jansen and W. van der Schee, Phys. Rev. D **94**, no. 6, 061901 (2016)
- [8] O. J. C. Dias, J. E. Santos and B. Way, arXiv:1702.07718

- [hep-th].
- [9] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
 - [10] K. Skenderis, Class. Quant. Grav. **19**, 5849 (2002)
 - [11] H. Elvang and M. Hadjiantonis, JHEP **1606**, 046 (2016)
 - [12] M. J. Duff and J. T. Liu, Nucl. Phys. B **554**, 237 (1999)
 - [13] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).
 - [14] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998)
 - [15] P. M. Chesler and L. G. Yaffe, JHEP **1407**, 086 (2014)
 - [16] P. Grandclement and J. Novak, Living Rev. Rel. **12**, 1 (2009)
 - [17] G. T. Horowitz, J. E. Santos and D. Tong, JHEP **1207**, 168 (2012)
 - [18] A. Donos and J. P. Gauntlett, JHEP **1404** (2014) 040 [arXiv:1311.3292 [hep-th]].